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PLANE WAVES ON A LIQUID SURFACE WITH HARMONIC PERTURBATIONS OF BOUNDARY COMPONENTS†

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Low-frequency asymptotic solutions are constructed for the wave problem with harmonic perturbations of parts of the boundary, neglecting surface tension. The asymptotic expansions are justified and estimates of the error of the solution are given.

This problem has been considered both without [1–4] and with [5, 6]‡ the inclusion of surface tension. In particular, a solution to the limiting problem with infinitely large dimensionless frequency (or the absence of gravity) was taken [3] as an initial approximation to the solution for the problem under consideration. The method of matched asymptotic expansions was then used at points where the boundary conditions change to fine-tune functions with large gradients.

Below we construct solutions bounded at infinity and with first-order discontinuities (with a finite jump) at points where the boundary conditions change. We present an iterative scheme that differs from the traditional one. This is due to the fact that the initial approximation for infinitely high frequencies gives unbounded free surface displacements at points where the boundary conditions change. If the solution is corrected by the deformed coordinate method, it becomes necessary to solve the wave problem for harmonic displacements of a semi-infinite part of the upper boundary of the liquid as a rigid whole. This corresponds to the displacement of an infinite body of liquid and requires the application of an infinite load to the upper boundary of the liquid, whereas in the original boundary-value problem such singularities do not occur. This was the motivation for developing an alternative iterative scheme which was free of these singularities.

1. STATEMENT OF THE PROBLEM

In the linear formulation with vanishingly small dissipative forces the two-dimensional steady wave motion of an ideal fluid with infinite depth with harmonic perturbations of part of its boundary is described by the following boundary-value problem [1]

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$$\partial \mathbf{U}/\partial t + \mu \mathbf{U} = -\rho^{-1}\nabla \mathcal{P}, \ \mathrm{div}\mathbf{U} = 0, \ \mathbf{U} = \nabla\Phi$$
$$\mathcal{P} = p + \rho g z - p_0, \ \mathcal{P} = -\rho[\partial\Phi/\partial t + \mu\Phi]$$
$$-\mathcal{P} + \rho g\zeta = -p_* = -\Pi(x)e^{i\omega t}, \ \partial\zeta/\partial t = U_z, \ z = 0, \ |x| > a$$
$$U_z = V_*(x, t) = \partial W_*(x, t)/\partial t, \ |x| \leq a, \ z = 0 \tag{1.1}$$
$$\lim_{r\to\infty} (\nabla\Phi) r^{1+\delta} = 0, \ \delta > 0, \ r = (x^2 + z^2)^{1/2}$$
$$\lim_{|x|\to\infty} |x|^{1/2+\delta}\mathbf{F} = 0, \ \mathbf{F} = \{\Phi, \varsigma, p_*\}, \ \delta > 0$$
$$\mathbf{F}(x, z, t + \frac{2\pi}{\omega}) = \mathbf{F}(x, z, t)$$

Here $\mathbf{U} = \{U_x, U_z\}$ is the velocity vector of the particles of the liquid, p is the hydrodynamic pressure, P is the dynamic part of the hydrodynamic pressure, $\rho g(-z)$ is the hydrostatic part of the hydrodynamic pressure, ζ is the upward displacement of the free surface, $V_*(x, t) = V_0(x)e^{i\omega t}$ is the oscillatory velocity of part of the upper boundary, $W_*(x, t) = W(x)e^{i\omega t}$ are the specified vertical displacements of points of the upper boundary of the liquid from the equilibrium state $p_*(p_0)$ is the external dynamic (static) pressure on the free surface of the liquid (and we set $p_* = 0$), ρ is the density of the liquid μ is a small dissipation coefficient ($\mu > 0$), $2a$ is the width of the excited part of the upper boundary of the liquid, and ω is the oscillation frequency. The origin of coordinates is taken to be the middle of the excited part of the boundary at the equilibrium position, the x axis is horizontal and the z axis is directed vertically upwards against the force of gravity.

We will seek a solution of problem (1.1) in the form

$$\Phi = \varphi e^{i\omega t}, \ \varsigma = \eta e^{i\omega t}, \ i\omega\eta = \partial\varphi/\partial z|_{z=0} \tag{1.2}$$

Here $\varphi(x, z)$ and $\eta(x)$ are amplitude functions for the velocity potential and upward displacement of the free boundary.

Substituting (1.2) into (1.1), we obtain the following boundary-value problem for φ

$$\Delta\varphi = 0$$
$$-\gamma\varphi + \partial\varphi/\partial z = -i\omega(\rho g)^{-1}\Pi, \ z = 0, \ |x| > a$$
$$\partial\varphi/\partial z = V_0(x) = i\omega W(x), \ z = 0, \ |x| \leq a \tag{1.3}$$
$$\{\varphi, \partial\varphi/\partial x, \partial\varphi/\partial z\} \to 0, \ x^2 + z^2 \to \infty$$
$$\gamma = (\omega^2 - i\omega\mu)/g.$$

2. REDUCTION OF THE ORIGINAL BOUNDARY-VALUE PROBLEM TO AN INTEGRAL EQUATION

We continue the function $\Pi(x)$ into the interior of the interval $|x| \leq a$, introducing the function $q(x)$

$$-\gamma\varphi + \frac{\partial\varphi}{\partial z} = -\frac{i\omega}{\rho g}q; \ q(x) = \begin{cases} \Pi(x), & |x| > a \\ Q(x), & |x| < a \end{cases} \tag{2.1}$$

In this problem $Q(x)$ is the amplitude function of the unknown excess pressure which deforms the excited part of the upper liquid boundary in the required manner.

We construct a solution of an auxiliary problem assuming that the function $Q(x)$ is known at this stage [7]

$$\left.\frac{\partial\varphi}{\partial z}\right|_{z=0} = -\frac{i\omega}{\rho g}q + (2\pi)^{-1/2}\gamma\frac{i\omega}{\rho g}\int_{-\infty}^{\infty} q(u)K(x-u)du$$

$$K(z) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}\frac{e^{-\xi z}}{|\xi|-\gamma}d\xi = \sqrt{\frac{2}{\pi}}G(\gamma|z|) - (2\pi)^{1/2}ie^{-i\gamma|z|}; \qquad (2.2)$$

$$G(u) = -[\cos(u)\mathrm{ci}(u) + \sin(u)\mathrm{si}(u)]$$

Here $\mathrm{ci}(u)$ is the integral cosine [8]. We consider the interval $|x| \leqslant a$. According to the boundary condition in (1.3) at $z=0$, $|x| \leqslant a$, and also using the fact that $q(x)=0$ when $|x|>a$, we obtain an integral equation in $Q(x)$

$$Q(x) + \rho g W(x) = -(2\pi)^{-1/2}\gamma\int_{-a}^{a} Q(u)K(x-u)du, \ |x| \leqslant a$$

We write integral equation (2.2) in the form

$$Q(x) + \rho g W(x) = UQ \equiv \gamma i\int_{-a}^{a} Q(u)e^{-i\gamma|x-u|}du + f(x)$$

$$f(x) = -\gamma\pi^{-1}\int_{-a}^{a} Q(u)G(\gamma|x-u|)du, \ |x| \leqslant a \qquad (2.3)$$

3. SOLUTION OF THE INTEGRAL EQUATION AT LOW FREQUENCIES

We replace the original integral equation (2.3) by the "approximation"

$$\overline{Q}(x) + \rho g W(x) = \overline{U}\ \overline{Q} \equiv \gamma i\int_{-a}^{a}\overline{Q}(u)e^{-i\gamma|x-u|}du \qquad (3.1)$$

We estimate the closeness of the integral operators in the original and "approximate" equation in the metric space C

$$\|(U-\overline{U})Q\| = \|f(x)\| \leqslant \pi^{-1}|\gamma|\|Q\|\max_x\int_{-a}^{a}|G(\gamma|x-u|)|du$$

Using integral representations of the functions $G(z)$, $\mathrm{ci}(z)$, $\mathrm{si}(z)$, we obtain

$$|G(z)| \leqslant |G(z_0)| \leqslant C + \pi/2 + |\ln(z_0)| + z_0^2/4$$

$$z_0 = \mathrm{Re}\, z > 0, \ C = 0.577$$

$$\|(U-\overline{U})Q\| \leqslant 2\pi^{-1}\varepsilon M\|Q\| \qquad (3.2)$$

$$M = |\ln(\varepsilon_0)| + C + \pi/2 + 1 + \varepsilon_0^2/3$$

$$\varepsilon = |\lambda|, \ \varepsilon_0 = \lambda_0, \ \lambda = \gamma a, \ \lambda_0 = \gamma_0 a$$

$$\gamma_0 = \mathrm{Re}\,\gamma = \omega^2/g$$

This shows that at low frequencies ($\varepsilon \to 0$, $\varepsilon_0 \to 0$) the integral operator $\overline{U}$ of the "approximate" equation is infinitesimally close in norm to the integral operator U of the original equation.

We solve the "approximate" integral equation (3.1). To this end we integrate Eq. (3.1) with respect to x. We obtain

$$R(x) + \rho g V(x) = \gamma i J_1 + \gamma i J_2 \tag{3.3}$$

$$R(x) = \int_0^x \overline{Q}(t)dt, \quad V(x) = \int_0^x W(t)dt$$

$$J_1 = \int_0^x \int_{-a}^t \overline{Q}(u)e^{-i\gamma(t-u)}du dt, \quad J_2 = \int_0^x \int_t^a \overline{Q}(u)e^{-i\gamma(u-t)}du dt$$

We transform the double integral J_1 as follows:

$$J_1 = \int_0^x \left[\frac{d}{dt} \int_{-a}^t \overline{Q}(u)e^{-i\gamma(t-u)} \frac{du}{-i\gamma} + \frac{\overline{Q}(t)}{i\gamma} \right] dt = -\frac{1}{i\gamma} \int_{-a}^x \overline{Q}(u)e^{-i\gamma(x-u)}du + \frac{R(x)}{i\gamma} +$$

$$+ \frac{1}{i\gamma} \int_{-a}^0 \overline{Q}(u)e^{i\gamma u}du$$

We similarly find J_2. We then multiply (3.3) by γ_2 and add the resulting equation to the result of differentiating (3.1) with respect to x. Using $Q' = R''$, $W' = V''$, we obtain

$$R'' - \gamma^2 R + \rho g \left[V'' + \gamma^2 V \right] = C_1$$

$$C_1 = \gamma^2 \int_0^a \left[\overline{Q}(u) - \overline{Q}(-u) \right] e^{-i\gamma u} du$$

From this we derive

$$\overline{Q}(x) = D_1 \operatorname{sh}(\gamma x) + D_2 \operatorname{ch}(\gamma x) - \rho g W(x) - 2\gamma \rho g \int_0^x W(\xi) \operatorname{sh}(\gamma(x-\xi)) d\xi \tag{3.4}$$

The constants D_1 and D_2 are found from the condition that $\overline{Q}(x)$ satisfies the integral equation (3.1). In particular, when $W(x) = W_0 = \text{const} > 0$, we have

$$\overline{Q}(x) = -\rho g W_0 \left[1 - 2D^{-1} \operatorname{ch}(\lambda) \right], \quad |x| \leq a \tag{3.5}$$

$$D(\lambda) = \operatorname{ch}(\lambda) - i \operatorname{sh}(\lambda), \quad \lambda = \gamma a$$

To estimate the error of the solution we write the original and "approximate" integral equations in operator form

$$(I - \overline{U})\overline{Q}(x) = -\rho g W(x); \quad (I - U)Q(x) = -\rho g W(x) \tag{3.6}$$

$$\overline{Q}(x) = (I - \overline{U})^{-1}(-\rho g W(x))$$

Then using the traditional estimate scheme [9] we obtain

$$\overline{Q} = (I - \overline{U})^{-1}(-\rho g W(x)) = (I - \overline{U})^{-1}\left[(I - \overline{U}) + (\overline{U} - U) \right] Q =$$

$$= Q + (I - \overline{U})^{-1}(\overline{U} - U)((Q - \overline{Q}) + \overline{Q}).$$

From this, and using estimate (3.2), we obtain

$$\|Q-\bar{Q}\| \leqslant \frac{\delta}{1-\delta}\|\bar{Q}\|, \ \delta = 2\pi^{-1}\varepsilon M\|(I-\bar{U})^{-1}\|$$

When $W(x)=W_0=\text{const}>0$ we find that $\|(I-\bar{U})^{-1}\|\leqslant 1$ using the last equation in (3.6) and the equations in (3.5). Finally, we obtain the following estimate for the error of the solution

$$\|Q-\bar{Q}\| \leqslant \alpha\rho g W_0/(1-\alpha), \ \alpha = 2\pi^{-1}\varepsilon M \tag{3.7}$$

The approximate solution $Q(x)$ of Eq. (3.1) can be made more precise using an iterative scheme based on the integral equation (2.3)

$$\begin{gathered} Q^{(n+1)}(x) = \bar{Q}(x) + f^{(n)}(x) + 2\gamma\int_0^x f^{(n)}(\xi)\operatorname{sh}(\gamma(x-\xi))d\xi \\ f^{(n)}(x) = -\frac{\gamma}{\pi}\int_{-a}^{a} Q^{(n)}(u)G(\gamma|x-u|)du, \ Q^{(0)}(x) = \bar{Q}(x) \end{gathered} \tag{3.8}$$

The function $Q(x)$ here is defined in (3.4), with

$$D_{1,2} = \delta_j(b_1 \mp b_2)/[\operatorname{ch}(\lambda) \pm i\operatorname{sh}(\lambda)], \ \delta_1 = -i, \ \delta_2 = 1$$

$$b_{1,2} = \mp\rho g\gamma i\int_{\mp a}^{0} W(\xi)[\operatorname{ch}(\gamma(a\pm\xi)) + i\operatorname{sh}(\gamma(a\pm\xi))]d\xi$$

It follows from estimate (3.2) that the iterative process converges, and we have the estimate

$$\|Q-Q^{(n+1)}\| \leqslant \alpha\|\bar{Q}\|/(1-\alpha) \tag{3.9}$$

In the $W(x)=W_0=\text{const}$ case with $n=1$ we find that

$$Q^{(2)}(x) = \rho g W_0\{-1-\pi^{-1}[x_2\ln(x_2) - x_1\ln(-x_1)] - \gamma a[2i + \pi^{-1}2(C-1)] + O(\xi)\} \tag{3.10}$$

$$x_{1,2} = \gamma(x \mp a)$$

$$Q(x) = Q^{(2)}(x) + O(\rho g W_0 \xi), \ \xi = |\lambda|^2|\ln(\lambda)|$$

Thus, the excess pressure at points where the boundary conditions change has a singularity of the type $\rho\ln\rho$, where ρ is the distance from the point where the boundary conditions change. The strength of this singularity is proportional to the dimensionless frequency γa.

4. DETERMINING THE FREE SURFACE AT LOW FREQUENCIES

Substituting expressions (3.5) and (3.7) into the final equation in (2.1), we determine the function $q(x)$. We then compute $\partial\varphi/\partial z|_{z=0}$ from (2.2), and use it in the final equation in (1.2) to find (for the case $W(x)=W_0=\text{const}>0$)

$$\eta(x) = -\frac{1}{\rho g}\frac{\gamma}{(2\pi)^{1/2}}\int_{-a}^{a}\rho g W_0[1-2D^{-1}\operatorname{ch}(\gamma u)]K(x-u)du + R_\eta \tag{4.1}$$

$$|R_\eta| \leqslant \frac{|\gamma|}{(2\pi)^{1/2}}\frac{1}{\rho g}|R_Q|\int_{-a}^{a}|K(x-u)|du, \ |R_Q| = |Q-\bar{Q}|$$

The integrals in (4.1) are evaluated by parts of the formulae

$$dF/dz = -G(z),\ dG/dz = F - z^{-1},\ F = d(G + \ln(z))/dz$$

$$F(z) = \int_0^\infty \frac{e^{-zt}}{t^2+1}dt,\ G(z) = \int_0^\infty \frac{te^{-zt}}{t^2+1}dt,\ \mathrm{Re}\, z > 0 \tag{4.2}$$

As a result we have

$$\eta(x) = Ae^{-ix_1} - A(2\pi i)^{-1}\psi_1(x) + B\psi_2(x) + R_\eta,\ x > a$$

$$A = W_0 D^{-1}(\lambda)\mathrm{sh}(\lambda);\ B = W_0(2\pi D(\lambda))^{-1}$$

$$\psi_1(x) = i[F(x_1) - F(x_2)] + G(x_1) + G(x_2)$$

$$\psi_2(x) = e^{\gamma x}[E_1(x_2) - E_1(x_1)] - e^{-\gamma x}[E_1(-x_2) - E_1(-x_1)] \tag{4.3}$$

$$E_1(z) = \int_z^\infty t^{-1}e^{-t}dt,\ |\arg(z)| < \pi$$

$$|R_\eta| \leqslant W_0\varepsilon\alpha(1+2M_1)/(1-\alpha),\ M_1 = C + \pi 2^{-1} + |\ln(2\varepsilon_0)| + \varepsilon_0^2$$

The quantities $D(\lambda)$, ε, ε_0 and λ occurring here are given by (3.5), (3.2) and (3.7).

From (4.3) with $x \to a^+$ we obtain the value of the amplitude function of the free surface at points where the boundary conditions change

$$\eta(a^+) = 2\lambda\pi^{-1}W_0[-\ln(2\lambda) + 1 - C] + O(W_0\xi)$$

Hence, the upper boundary of the liquid undergoes a finite first-order discontinuity at the points where the boundary conditions change $x = \pm a$, the magnitude of the jump being proportional to the dimensionless frequency.

5. AN ESTIMATE OF THE DISCREPANCY OF THE BOUNDARY CONDITION AT LOW FREQUENCIES IN THE ORIGINAL PROBLEM

We shall also estimate the discrepancy in the boundary condition in (1.1) when $z = 0$, $|x| \leqslant a$, i.e. compute the amplitude function $\eta(x)$ for the deformation of the upper boundary of the liquid at $|x| \leqslant a$ and compare it with the specified value $\eta(x) = W_0$. We shall show that

$$\eta(x) = W_0[1 + O(\xi)]. \tag{5.1}$$

From (4.1) and (2.2) at $|x| \leqslant a$ we evaluate the integrals and obtain

$$\eta(x) = W_0\{2 - e^{-ix_2} - e^{ix_1} + iD^{-1}(\lambda)D_1(\gamma x)[1 - e^{-ix_2}] + iD^{-1}(\lambda)D_2(\gamma x)[1 - e^{ix_1}] - 1 +$$
$$+\pi^{-1}[F(x_2) - F(-x_1)] - 2^{-1}\mathrm{ch}(\gamma x) + (2\pi)^{-1}\mathrm{ch}(\lambda)[F(x_2) + F(-x_1)] + (2\pi)^{-1}\mathrm{sh}(\lambda)[G(-x_1) +$$
$$+\ln(-x_1) + G(x_2) + \ln(x_2)] + O(\xi)\} \tag{5.2}$$

From this, using the representations of $F(z)$ and $G(z)$ when $|z| \to 0$, we obtain (5.1).

6. SOLUTION OF THE INTEGRAL EQUATION FOR FIXED BOUNDED FREQUENCIES

We shall show that for all $\lambda = \gamma a$ of sufficiently small absolute magnitude ($|\lambda| < \lambda_*$), the integral operator U specified by the right-hand side of Eq. (2.3) is a contraction operator.

We assume that $W(x)$ belongs to the space L^2 and we seek a solution of (2.3) among functions in L^2.

Let Q_1 and Q_2 be two arbitrary functions in L^2. We shall show that a positive constant $\alpha < 1$ exists such that

$$\rho(UQ_1, UQ_2) \leq \alpha\rho(Q_1, Q_2), \quad \rho(y_1, y_2) = \|y_1 - y_2\|$$

Putting $\xi = Q_1 - Q_2$, $y_1 = UQ_1$, $y_2 = UQ_2$, we have

$$|y_1(x) - y_2(x)| \leq |\gamma| \int_{-a}^{a} |\xi(u)| du + \pi^{-1} |\gamma| \int_{-a}^{a} |\xi(u)| |G(\gamma|x - u|)| du$$

From this, using the Cauchy–Bunyakovskii inequality and the estimate for $G(z)$ from (3.2), we obtain

$$|y_1(x) - y_2(x)| \leq |\gamma| \left\{ (2a)^{1/2} + \pi^{-1} \left[\int_{-a}^{a} G^2(\gamma_0 |x - u|) du \right]^{1/2} \right\} \|\xi\|$$

Squaring both sides of this equation and integrating, we have

$$\|y_1 - y_2\| \leq |\lambda| B \|\xi\|, \quad B = 4 + 2^{3/2}\pi^{-1} \int_{-1}^{1} f^{1/2}(s) ds + \pi^{-2} \int_{-1}^{1} f(s) ds$$

$$f(s) = \int_{-1}^{1} G^2(\lambda_0 |s - t|) dt, \quad \lambda_0 = \gamma_0 a$$

It follows from this that when $|\gamma a| < B^{-1} = \lambda_*$, the operator U is a contraction operator. Thus, by the Banach theorem [9], we have proved the existence of uniqueness of the solution of Eq. (2.3), and this solution can be obtained by the method of successive approximations. One can take any function in L^2 to be the initial function. Here uniqueness means the uniqueness "up to a transition to an equivalent function" [9].

In the case when $W(x) = W_0 = \text{const} > 0$, taking the initial function to be $Q = (0)$, we find

$$Q^{(1)}(x) = -\rho g W_0$$

$$Q^{(2)}(x) = -\rho g W_0 \left\{ 1 - e^{-ix_2} - e^{ix_1} + 1 + \pi^{-1} \left[(-F(0) + F(x_2)) + (F(-x_1) - F(0)) \right] \right\} - \rho g W_0$$

$$F(z) = \mathrm{ci}(z)\sin(z) - \mathrm{si}(z)\cos(z), \quad \mathrm{Re}\, z > 0$$

Expanding the functions in series, we obtain

$$Q^{(2)}(x) = \rho g W_0 \left\{ -1 - \pi^{-1} \left[x_2 \ln(x_2) - x_1 \ln(-x_1) \right] - \lambda \left[2i + 2\pi^{-1}(C - 1) \right] + O(\xi) \right\}$$

which is identical with the representation (3.10) for $Q^{(2)}(x)$ obtained by other methods.

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